Index theory for linear self-adjoint operator equations and nontrivial solutions for asymptotically linear operator equations(II)*

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Abstract

Reference [1] established an index theory for a class of linear selfadjoint operator equations covering both second order linear Hamiltonian systems and first order linear Hamiltonian systems as special cases. In this paper based upon this index theory we construct a new reduced functional to investigate multiple solutions for asymptotically linear operator equations by Morse theory. The functional is defined on an infinite dimensional Hilbert space, is twice differentiable and has a finite Morse index. Investigating critical points of this functional by Morse theory gives us a unified way to deal with nontrivial solutions of both asymptotically second order Hamiltonian systems and asymptotically first order Hamiltonian systems.

Key Words: Linear selfadjoint operator equations, index theory, asymptotically linear operator equations, multiple solutions, reduced functional, Morse theory.

1 Introduction and main results

Let X be a real separable infinite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $A : D(A) \subset X \to X$ be a unbounded linear self-adjoint operator with domain $D(A)$ satisfying $\sigma(A) = \sigma_d(A)$. We investigate the following equation

$$Ax - \Phi'(x) = \theta, \tag{1.1}$$

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where $\Phi \in C^1(X)$ and $\Phi'(x)$ is the derivative of Φ with respect to x in X. This equation covers both second order Hamiltonian systems and first order Hamiltonian systems as special cases. We will construct a new reduced functional to investigate (1.1) by Morse theory. The functional is defined on an infinite subspace of X, is twice differentiable and has a finite dimensional Morse index at its any critical point. The main result is the following theorem.

Theorem 1.1. Assume that

(i) $\Phi''(x)$ exists and is bounded for $x \in X$, $\Phi'(\theta) = \theta$, $\Phi \in C^2(V)$ with $V := \mathrm{D}(|A|^{\frac{1}{2}})$;

(ii) there exists $B_1,\ B_2 \in L_s(X)$ satisfying $i_A(B_1) = i_A(B_2)$, $\nu_A(B_2) = 0$ and $B : X \to L_s(X), C : X \to X$ such that

$$\Phi'(x) = B(x)x + C(x) \ \text{ for any } \ x,$$
$$B_1 \leq B(x) \leq B_2, \quad C(x) \ \text{ is bounded;}$$

(iii)with $B_0 := \Phi''(\theta)$ we have

$$i_A(B_1) \notin [i_A(B_0), i_A(B_0) + \nu_A(B_0)].$$

Then (1.1) has a nontrivial solution $x = x_0$.

Under the further assumption that

(iv) $\nu_A(B_0) = 0$ and $|i_A(B_1) - i_A(B_0)| \geq \nu_A(\Phi''(x_0))$, (1.1) has two nontrivial solutions.

In the theorem we used notations $(i_A(B), \nu_A(B))$ concerning the linear selfadjoint operator equation

$$Ax - Bx = 0 \tag{1.2}$$

for any $B \in L_s(X)$, which will be defined as follows.

Definition 1.2 For any $B \in L_s(X)$, we define

$$\nu_A(B) = \dim\ker(A - B). \tag{1.3}$$

$\nu_A(B)$ is called the nullity of B.

Definition 1.3 For any $B_1, B_2 \in L_s(X)$ with $B_1 < B_2$, we define

$$I_A(B_1, B_2) = \sum_{\lambda \in [0,1)} \nu_A((1-\lambda)B_1 + \lambda B_2); \tag{1.4}$$

and for any $B_1, B_2 \in L_s(X)$ we define

$$I_A(B_1, B_2) = I_A(B_1, kI) - I_A(B_2, kI) \tag{1.5}$$

where $I : X \to X$ is the identity map and $kI > B_1, kI > B_2$ for some real number $k > 0$. We call $I_A(B_1, B_2)$ the relative Morse index between B_1 and B_2.

Let $B_0 \in L_s(X)$ be fixed and let $i_A(B_0)$ be a prescribed integer associated with B_0.

Definition 1.4 For any $B \in L_s(X)$ we define

$$i_A(B) = i_A(B_0) + I_A(B_0, B). \tag{1.6}$$

As in [1] we call $i_A(B)$ the index of B and $i_A(B_0)$ is called initial index. Generally, the initial index can be any prescribed integer and the index $i_A(B)$ also depends on B_0 and the initial index. Let X_1 be a nontrivial subspace of X. For $B_1, B_2 \in \mathcal{L}_s(X)$ we write $B_1 \le B_2$ with respect to X_1 if and only if $(B_1x, x) \le (B_2x, x)$ for any $x \in X_1$; we write $B_1 < B_2$ with respect to X_1 if and only if $(B_1x, x) < (B_2x, x)$ for any $x \in X_1\backslash\{\theta\}$. If $X_1 = X$ we just write $B_1 \le B_2$ or $B_1 < B_2$.

Theorem 1.5 (i) For any $B, B_1, B_2 \in L_s(X)$, $\nu_A(B) \in \mathbf{N}, I_A(B_1, B_2) \in \mathbf{Z}$ and $i_A(B) \in \mathbf{Z}$ are well-defined;

(ii) For any $B_1, B_2 \in L_s(X)$, $I_A(B_1, B_2) = i_A(B_2) - i_A(B_1)$, and if $B_1 < B_2$ with respect to $\mathrm{Ker}(A - (1-\lambda)B_1 - \lambda B_2) \ne \{\theta\}$ for $t \in [0, 1)$, then (1.4) holds;

(iii) For any $B_1, B_2, B_3 \in L_s(X)$, $I_A(B_1, B_2) + I_A(B_2, B_3) = I_A(B_1, B_3)$;

(iv) For any $B_1, B_2 \in L_s(X)$, if $B_1 \le B_2$, then $i_A(B_1) \le i_A(B_2), \nu_A(B_1) + i_A(B_1) \le \nu_A(B_2) + i_A(B_2)$; if $B_1 < B_2$ with respect to $\mathrm{Ker}(A - B_1)$, then $\nu_A(B_1) + i_A(B_1) \le i_A(B_2)$.

(v) If there exists $B_0 \in L_s(X)$ such that $\sum_{\lambda<0} \nu_A(B_0 + \lambda I) < +\infty$, we will choose this integer for $i_A(B_0)$. Then the index defined by Definition 1.3 satisfies

$$i_A(B) = \sum_{\lambda<0} \nu_A(B + \lambda I). \tag{1.7}$$

In [1] an index theory for $Ax + Bx = \theta$ was established by the concept of relative Morse index and dual variational methods. Here we discuss (1.2) instead only because the new form will bring convenience to the proof of Theorem 1.1 as will be seen in Sections 4-5. In [1] it was assumed that A satisfies the following condition:

(A)*:* $A : Y \to X$ *is linear bounded, symmetric i.e.* $(Ax, y) = (x, Ay)$ *for any* $x, y \in Y$*,* $R(A)$ *is closed in* X *and* $X = R(A) \bigoplus ker(A)$*, where* X *is a real separable infinite dimensional Hilbert space,* $Y \subset X$ *is a Banach space and the embedding* $Y \hookrightarrow X$ *is compact.*

In order to prove Theorem 1.5 we first prove the following proposition.

Proposition 1.6 $A : D(A) \subset X \to X$ is selfadjoint and $\sigma(A) = \sigma_d(A)$ if and only if A satisfies condition (A).

Proof. Sufficiency: Denote all the eigenvalues of A with multiplicities by $\{\lambda_j\}_{j=-\infty}^{\infty}$ satisfying $\lambda_j \le \lambda_{j+1} \forall j$ and $\lambda_j \to \pm\infty$ as $j \to \pm\infty$. There is a unit orthogonal basis $\{e_j\}$ of X such that $X = \{\sum_{j=-\infty}^{\infty} c_j e_j | \sum c_j^2 < \infty\}$ and $D(A) = \{\sum_{j=-\infty}^{\infty} c_j e_j | \sum_{j=-\infty}^{\infty}(1+\lambda_j^2)c_j^2 < \infty\}$. For any $x = \sum_{j=-\infty}^{\infty} c_j e_j \in X$, because $x_n := \sum_{j=-n}^{n} c_j e_j \in D(A)$ and $x_n \to x$ in X. Thus, $D(A)$ is dense in X. In order to prove the adjointness of A we need only $D(A^*) \subset D(A)$. In fact, assume $x = \sum_{j=-\infty}^{\infty} c_j e_j \in D(A^*)$. By definition there exists a constant $C > 0$ such that $|(Ay, x)| \le C||y||$ for any $y \in D(A)$. If we choose $y = \sum_{j=-n}^{n} \lambda_j c_j e_j \in D(A)\ \forall\ n$, then $\sum_{j=-n}^{n} \lambda_j^2 c_j^2 \le C^2$. Thus $x \in D(A)$.

Necessity: Because $\sigma(A) = \sigma_d(A)$, by definition, $D(A) = \{x \in X | \Sigma_{\lambda\in\sigma_d(A)}(1+\lambda^2)\|E(\{\lambda\})x\|^2 < \infty\}$; and $\forall \lambda \in \sigma_d(A)$, λ is isolated and $E(\{\lambda\})X = ker(A-\lambda I)$, the embedding $(D(A), \|\cdot\|_G) \hookrightarrow X$ is compact. To finish the proof, we prove $R(A)$ is closed. In fact, any $x \in X$ satisfies $x = \sum_{\lambda\in\sigma_d(A)} E(\{\lambda\})x$. If $x \perp ker(A)$, then $x = \sum_{\lambda\in\sigma_d(A)\setminus\{0\}} E(\{\lambda\})x$ and $\sum_{\lambda\in\sigma_d(A)\setminus\{0\}} \frac{1}{\lambda} E(\{\lambda\})x \in D(A)$. This means that $R(A) = (kerA)^{\perp}$ is closed. ∎

Proof of Theorem 1.5 We give two proofs.

Step 1: Set $A_1 := -A$. Because A is selfadjoint and $\sigma(A) = \sigma_d(A)$, so does A_1. By Proposition 1.6, A_1 satisfies condition (A). From [1, Definitions 3.1.1, 3.1.2 and 3.1.3] $(i_{A_1}(B), \nu_{A_1}(B))$ is defined. Denote $(i_{A_1}(B), \nu_{A_1}(B))$ by $(i_A(B), \nu_A(B))$. Then (1.3), (1.4), (1.5) and (1.6) are satisfied. And [1, Propositions 3.1.4 and 3.1.5, and Lemma 3.2.1] imply all the conclusions of Theorem 1.4.

Step 2: Because $\sigma(A) = \sigma_d(A)$, every eigenvalue is isolated and corresponds to a finite dimensional subspace of eigenvectors. Let $-\mu \notin \sigma(A)$. Then $A + \mu I$ is invertible. For any $B \in L_s(X)$ and such a $\mu > 0$ large enough satisfying $B + \mu I > I$, the following bilinear form

$$\psi_{A,\mu;B}(x, y) = ((B + \mu I)^{-1}x, y) - ((A + \mu I)^{-1}x, y), \quad \forall x, y \in X$$

has finite Morse index $m^-_{A,\mu}(B)$ and finite Morse nullity $m^0_{A,\mu}(B)$. Because the first term in $\psi_{A,\mu;B}$ is the same with the second term in $\psi_{A,B|B_0}$ defined by (3.5) in [1] when $B_0 = \mu I$, and the left terms in both these two bilinear forms do not depend on B, all the associated conclusions of [1, Theorem 3.2.4] hold, and from which in a way similar to the proofs of [1, Propositions 3.1.4 and 3.1.5] we can complete the proof. ∎

Many authors investigated (1.1) and its special cases-first order or second order Hamiltonian systems. We refer to [2,3,4,5,6,7,8,17] for references. Reference [1] investigated (1.1) by Morse theory only in the case $\sigma(A)$ is bounded from below and the result cannot apply to first order Hamiltonian systems. However, Theorem 1.1 applies to both second order Hamiltonian systems and first order Hamiltonian systems respectively. The paper will be organized in the following

way. In Section 2 as applications of Theorem 1.1 we investigate first order and second order asymptotically Hamiltonian systems. In Section 3, we construct the mentioned reduced functional. In Section 4 we investigate properties of the Morse index of the functional at any critical point. And in the last section we prove Theorem 1.1 by Morse theory.

2 Applications of Theorem 1.1: Nontrivial solutions for asymptotically linear Hamiltonian systems

In this section as applications of Theorem 1.1 we investigate nontrivial solutions of both first order Hamiltonian systems and second order Hamiltonian systems satisfying various boundary value conditions.

2.1 Hamiltonian systems satisfying Bolza boundary value conditions

As in [1, Section 3.4] we are interested in the index theory for the following Hamiltonian system

$$-J\dot{x} - B(t)x = 0 \tag{2.1}$$

$$x_1(0)\cos\alpha + x_2(0)\sin\alpha = 0 \tag{2.2}$$

$$x_1(1)\cos\beta + x_2(1)\sin\beta = 0 \tag{2.3}$$

where $B \in L^\infty((0,1); GL_s(\mathbf{R}^{2n})), 0 \le \alpha < \pi$ and $0 < \beta \le \pi, x = (x_1, x_2) \in \mathbf{R}^n \times \mathbf{R}^n$. Define $X := L^2([0,1]; \mathbf{R}^{2n})$, $D(A) = \{x \in H^1([0,1]; \mathbf{R}^{2n}) | x \text{ satisfies } (2.2-2.3)\}$, $(Ax)(t) = -J\dot{x}(t)$ for any $x \in D(A)$, and $(\bar{B}x)(t) = B(x)x(t)$ for $x \in L^2([0,1], \mathbf{R}^{2n})$. Then $D(A)$ with the graph norm $\|\cdot\|_G$ is a Banach space and the embedding from $D(A)$ to X is compact. It is also easy to check that A is symmetric i.e. $(Ax, y) = (x, Ay)$ for any $x, y \in D(A)$. As proved in [1] there hold that $R(A)$ is closed in X and $X = R(A) \oplus \ker(A)$. So we have the following definitions and properties.

Definition 2.1[1, Definition 3.4.4] For any $B \in L^\infty((0,1); GL_s(\mathbf{R}^{2n}))$, we define

$$\nu^f_{\alpha,\beta}(B) := \dim\ker(A - \bar{B}),$$
$$i^f_{\alpha,\beta}(\mathrm{diag}\{0, I_n\}) := i^s_{I_n,\alpha,\beta}(0),$$
$$i^f_{\alpha,\beta}(B) := i^f_{\alpha,\beta}(\mathrm{diag}\{0, I_n\}) + I^f_{\alpha,\beta}(\mathrm{diag}\{0, I_n\}, B);$$

and

$$I^f_{\alpha,\beta}(B_1, B_2) = \sum_{\lambda\in[0,1)} \nu^f_{\alpha,\beta}((1-\lambda)B_1 + \lambda B_2) \quad \text{as} \quad B_1 < B_2,$$
$$I^f_{\alpha,\beta}(B_1, B_2) = I^f_{\alpha,\beta}(B_1, kI) - I^f_{\alpha,\beta}(B_2, kI) \quad \text{for every} \quad B_1, B_2 \quad \text{with} \quad kI > B_1, kI > B_2$$

where for any $B \in L^\infty([0,1]; GL_s(\mathbf{R}^n))$, $(i^s_{I_n,\alpha,\beta}(B), \nu^s_{I_n,\alpha,\beta}(B)) \in \mathbf{N} \times \{0,1,2,\cdots,n\}$ will be defined in Definition 2.5.

Proposition 2.2[1, Proposition 3.4.2] We have the following properties:

(i) For any $B \in L^\infty((0,1); GL_s(\mathbf{R}^{2n}))$, $\nu^f_{\alpha,\beta}(B)$ is the dimension of the solution subspace of system (2.1-2.3) and

$$(i^f_{\alpha,\beta}(B), \nu^f_{I_n,\alpha,\beta}(B)) \in \mathbf{Z} \times \{0,1,2,\cdots,n\}.$$

(ii) For any $B_1, B_2 \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$, if $B_1 \le B_2$, then $i^f_{\alpha,\beta}(B_1) \le i^f_{\alpha,\beta}(B_2)$ and $i^f_{\alpha,\beta}(B_1) + \nu^f_{\alpha,\beta}(B_1) \le i^f_{\alpha,\beta}(B_2) + \nu^f_{\alpha,\beta}(B_1)$; if $B_1 < B_2$ then $i^f_{\alpha,\beta}(B_1) + \nu^f_{\alpha,\beta}(B_1) \le i^f_{\alpha,\beta}(B_2)$.

(iii) For any $B \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^n))$, there holds

$$(i^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\}), \nu^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\})) = (i^s_{I_n,\alpha,\beta}(B), \nu^s_{I_n,\alpha,\beta}(B)).$$

Here for any $B_1,\ B_2 \in L^\infty([0,1]; GL_s(\mathbf{R}^{2n}))$, we write $B_1 \le B_2$ if $B_1(t) \le B_2(t)$ for a.e. $t \in [0,1]$; and we write $B_1 < B_2$ if $B_1 \le B_2$ and $B_1(t) < B_2(t)$ for t belonging to a subset of $(0,1)$ with nonzero measure. If $B_1 \le B_2$ then $\bar{B}_1 \le \bar{B}_2$; and if $B_1 < B_2$ then $\bar{B}_1 < \bar{B}_2$ with respect to $\mathrm{Ker}(A - \bar{B}_1)$. So Proposition 2.2(ii) follows from Theorem1.5(iv) directly.

We now use this index theory to discuss the solvability of the following Hamiltonian system (2.2-2.3) and

$$-J\dot{x} - H'(t,x) = \theta, \tag{2.4}$$

where $H : [0,1] \times \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is differentiable and $H'(t,x)$ is the gradient of H with respect to x.

Let $H''(t,x)$ denote the second derivative of $H(t,x)$ with respect to x. We have the following theorem.

Theorem 2.3. Assume that

(i) $H''(t,x)$ is continuous and is bounded for $(t,x) \in [0,1] \times \mathbf{R}^{2n}$, and $H'(t,\theta) = \theta$;

(ii) there exists $B_1,\ B_2 \in L^\infty([0,1]; GL_s(\mathbf{R}^{2n}))$ satisfying $i^f_{\alpha,\beta}(B_1) = i^f_{\alpha,\beta}(B_2)$, $\nu^f_{\alpha,\beta}(B_2) = 0$ and

$$B_1(t) \le H''(t,x) \le B_2(t) \quad \forall (t,x) \in [0,1] \times \mathbf{R}^{2n} \quad with \quad |x| \ge r > 0;$$

(iii)with $B_0 := H''(\cdot,\theta)$ we have

$$i^f_{\alpha,\beta}(B_1) \notin [i^f_{\alpha,\beta}(B_0), i^f_{\alpha,\beta}(B_0) + \nu^f_{\alpha,\beta}(B_0)].$$

Then (2.4)(2.2-2.3) has a nontrivial solution.

Under the further assumption that

(iv) $\nu^f_{\alpha,\beta}(B_0) = 0$ and $|i^f_{\alpha,\beta}(B_1) - i^f_{\alpha,\beta}(B_0)| \ge n$, (2.4)(2.2-2.3) has two nontrivial solutions.

Define

$$\Phi(x) = \int_0^1 H(t, x(t))dt \quad \forall x \in X \tag{2.5}$$

It is easy to check that (2.4)(2.2-2.3) is equivalent to (1.1). So in order to prove Theorem 2.3 by Theorem 1.1 we only need the following lemma.

Lemma 2.4 (i) $\Phi \in C^2(V)$;

(ii) Under assumption(ii) of Theorem 2.3, there holds

$$H'(t, x) = B(t, x)x + C(t, x)$$

$$B_1(t) - \epsilon_1 I_{2n} \leq B(t, x) \leq B_2(t) + \epsilon_1 I_{2n}$$

where for any $x \in X$, $B(\cdot, x(\cdot)) \in L_s(X)$ $\epsilon_1 > 0$ satisfying $i^f_{\alpha,\beta}(B_1 - \epsilon_1 I_{2n}) = i^f_{\alpha,\beta}(B_1) = i^f_{\alpha,\beta}(B_2)$, $\nu^f_{\alpha,\beta}(B_2 + \epsilon_1 I_{2n}) = 0$, and $C(\cdot, x(\cdot)) \in X$ is uniformly bounded.

Proof. (i) From [1] it follows that $\sigma(A) = \sigma_d(A) \subset \mathbf{R}$, and $\sigma_d(A)$ is unbounded from both above and from bellow. In fact, if $\lambda \neq 0$ is an eigenvalue of A with an eigenvector $e^{\lambda J t}c$ for $c \in \mathbf{R}^{2n}$, then for any integer k, $\lambda + 2k\pi$ is also an eigenvalue with the eigenvector $e^{(\lambda+2\pi)Jt}c$. Note that $V = \{\sum_{j=-\infty}^{\infty} c_j e_j | \sum_{j=-\infty}^{\infty}(1 + |\lambda_j|)c_j^2 < \infty\}$, where $\{e_j\}$ is an orthonormal basis of X, $\{\lambda_j\}$ is all the eigenvalues of A with multiplicities and $\lambda_j \leq \lambda_{j+1}$ $\forall j$. For any $x, x_0 \in V$ and $u = \sum_{j=-\infty}^{\infty} c_j e_j, v = \sum_{j=-\infty}^{\infty} c_j' e_j \in V$ satisfying $||u||_V \leq 1$ and $||v||_V \leq 1$, we have $|c_j| \leq 1$, $|c_j'| \leq 1$, and by assumption (ii) $\exists M > 0$ such that $||H''(\cdot, x(\cdot))u|| \leq M||u||$ for any $x, u \in X$. Hence,

$$\begin{aligned}
||\Phi_V''(x) - \Phi_V''(x_0)||_V &= \sup_{||u||_V \leq 1, ||v||_V \leq 1} |\int_0^1 ((H''(t, x(t)) - H''(t, x_0(t)))u(t), v(t))dt| \\
&\leq \sum_{i,j=-n}^{n} |\int_0^1 ((H''(t, x(t)) - H''(t, x_0(t)))e_j(t), e_i(t))dt| + 4M(\frac{1}{|\lambda_{-n-1}|} + \frac{1}{|\lambda_{n+1}|})
\end{aligned}$$

Because $\lambda_j \to \pm\infty$ as $j \to \pm\infty$ and from Theorem 4 in page 97 of Ekeland's book[13] it follows that for fixed i, j as $x \to x_0$ in X

$$\int_0^1 ((H''(t, x(t)) - H''(t, x_0(t)))e_j(t), e_i(t))dt \to 0;$$

we obtain

$$||\Phi_V''(x) - \Phi_V''(x_0)||_V \to 0$$

as $x \to x_0$ in V.

(ii) From assumption (ii) there exists $\epsilon_1 > 0$ such that $i^f_{\alpha,\beta}(B_1 - \epsilon_1 I_n) = i^f_{\alpha,\beta}(B_1)$, $i^f_{\alpha,\beta}(B_2 + \epsilon_1 I_n) + \nu^f_{\alpha,\beta}(B_2 + \epsilon_1 I_n) = i^f_{\alpha,\beta}(B_2)$. And we can choose $\delta \in (0,1)$ such that

$$-\epsilon_1/2 \le \delta B_1 \le \delta B_2 \le \epsilon_1/2, \quad \text{and} \quad \delta M < \frac{1}{2}\epsilon_1.$$

Define

$$\begin{aligned} B(t,x) \quad &= \int_0^1 H''(t,\theta x)d\theta \quad \text{if } |x| \ge r/\delta, \\ &= B_1(t) \text{ otherwise;} \end{aligned}$$

and $C(t,x) = H'(t,x) - B(t,x)x$. Then for any $x \in X, B(\cdot, x(\cdot)) \in L_s(X), B_1 - \epsilon I_{2n} \le B(\cdot, x(\cdot)) \le B_2 + \epsilon I_{2n}$ and $C(\cdot, x(\cdot)) \in X$ is bounded uniformly for $x \in X$. The proof is complete. ∎

Remark. In the special case $\alpha = 0, \beta = \pi$, Theorem 2.3 was given in [10]. However the proof there is not correct because generally the integral functional defined by (2.5) is not twice differentiable in $L^2([0,1]; \mathbf{R}^{2n})$ even we assume $H''(t,x)$ is continuous and bounded for $(t,x) \in [0,1] \times \mathbf{R}^{2n}$.

Theorem 1.1 can also be used to investigate second order Hamiltonian systems satisfying Sturm-Liouville boundary value conditions. Recall that an index theory has been established in [1] for the following system:

$$-\ddot{x} - B(t)x = 0 \tag{2.6}$$

$$x(0)\cos\alpha - x'(0)\sin\alpha = 0 \tag{2.7}$$

$$x(1)\cos\beta - x'(1)\sin\beta = 0 \tag{2.8}$$

where $0 \le \alpha < \pi$ and $0 < \beta \le \pi$.

Definition 2.5[1, Definition 2.3.2 and Proposition 2.3.3] We define

$$\begin{aligned} &\nu^s_{\alpha,\beta}(B) \ \text{ is the dimension of the solution subspace of } \ (2.6-2.8) \\ &i^s_{\alpha,\beta}(B) := \sum_{\lambda<0} \nu^s_{\alpha,\beta}(B + \lambda I_n). \end{aligned}$$

As before $(i^s_{\alpha,\beta}(B), \nu^s_{\alpha,\beta}(B))$ has useful properties, which can be found in [1]. This index can be used to investigate the following nonlinear Hamiltonian system (2.7-2.8) and

$$-\ddot{x} - V'(t,x) = 0, \tag{2.9}$$

where $V : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $V'(t,x)$ denotes the gradient of $V(t,x)$ with respect to x. Let $V''(t,x)$ denote the second derivative of $V(t,x)$ with respect to x. We have the following theorem.

Theorem 2.6 Assume that

(i) $V''(t,x)$ is continuous and is bounded for $(t,x)\in[0,1]\times\mathbf{R}^{2n}$, and $H'(t,\theta)=\theta$;

(ii) there exist $B_1,\ B_2\in L^\infty([0,1];GL_s(\mathbf{R}^n))$ satisfying $i^s_{\alpha,\beta}(B_1)=i^s_{\alpha,\beta}(B_2)$, $\nu^s_{\alpha,\beta}(B_2)=0$ and

$$B_1(t)\le V''(t,x)\le B_2(t)\ \ \forall (t,x)\in[0,1]\times\mathbf{R}^n\ \ with\ \ |x|\ge r>0;$$

(iii)with $B_0:=V''(\cdot,\theta)$ we have

$$i^s_{\alpha,\beta}(B_1)\notin[i^s_{\alpha,\beta}(B_0),i^s_{\alpha,\beta}(B_0)+\nu^s_{\alpha,\beta}(B_0)].$$

Then (2.9)(2.7-2.8) has a nontrivial solution.

Under the further assumption that

(iv) $\nu^s_{\alpha,\beta}(B_0)=0$ and $|i^s_{\alpha,\beta}(B_1)-i^s_{\alpha,\beta}(B_0)|\ge n$, (2.9)(2.7-2.8) has two nontrivial solutions.

Proof Define $x_1=x,x_2=-\dot{x},x=(x_1,x_2)$ and $H(t,x)=V(t,x_1)+\frac{1}{2}|x_2|^2$. Then (2.9)(2.7-2.8) is equivalent to (2.4)(2.2-2.3). Under assumption (ii) we have $V'(t,x_1)=B(t,x_1)x_1+C(t,x_1)$ as before. Because $i^s_{\alpha,\beta}(B)=i^f_{\alpha,\beta}(\mathrm{diag}\{B,I_n\})$, the result follows. ∎

Remark For the special case $\alpha=0,\beta=\pi$ this theorem was obtained in [1, Theorem 2.3.7], and [9, Theorem 3.3] by different methods.

2.2 Hamiltonian systems satisfying periodic boundary value conditions

Consider the following linear system

$$\begin{aligned}&-J\dot{x}-B(t)x=0\\&x(1)=Px(0)\end{aligned}\tag{2.10}$$

where $P\in Sp(2n)$ is prescribed. Define $X:=L^2([0,1];\mathbf{R}^{2n}),D(A):=\{x:\in H^1([0,1];\mathbf{R}^{2n})|x$ satisfies (2.10)$\}$. Then the embedding from $D(A)$ to X is compact. Define $(Ax)(t):=-J\dot{x}(t)$ for every $x\in D(A)$. Similar to Proposition 7 in page 22 of Ekeland's book[13], for the given $P\in Sp(2n)$ there exists $\lambda\in\mathbf{R}$ such that $(e^{J\lambda}-P)c=0$ for some $c\neq0$. So λ is an eigenvalue of A with an eigenvector $e^{Jt\lambda}c$. We can check $\lambda+2k\pi$ is also an eigenvalue of A with the eigenvector $e^{Jt(\lambda+2\pi)}c$. As in Lemma 2.4 A is selfadjoint and $\sigma(A)=\sigma_d(A)$ is unbounded from both bellow and above.

Choose $i^f_P(0):=i_P(I_{2n})$ defined by Definition 2.2 in [11]. We have the following definition.

Definition 2.7[1, Definition 3.5.1] For any $B\in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$, we define

$$\begin{aligned}&\nu^f_P(B)=dim\ker(A-B),\\&i^f_P(B)=i^f_P(0)+I^f_P(0,B);\end{aligned}$$

and

$$I_P^f(B_1,B_2)=\sum_{\lambda\in[0,1)}\nu_P^f((1-\lambda)B_1+\lambda B_2)\ \text{ as }\ B_1<B_2,$$
$$I_P^f(B_1,B_2)=I_P^f(B_1,kid)-I_P^f(B_2,kid)\ \text{ for every }\ B_1,B_2\ \text{ with }\ kI>B_1,kI>B_2.$$

From Theorem 1.5 we have the following proposition.

Proposition 2.8[1, Proposition 3.5.2]. (i)For any $B\in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$, we have $\nu_P^f(B)\in\{0,1,2,\cdots,2n\}$.

(ii) For any $B_1,B_2\in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$ satisfying $B_1<B_2$, we have $i_P^f(B_1)+\nu_P^f(B_1)\le i_P^f(B_2)$.

We now discuss the solvability of the following nonlinear system (2.4) (2.10)

$$-J\dot{x}-H'(t,x)=0$$
$$x(1)=Px(0)$$

where $H:[0,1]\times\mathbf{R}^{2n}\to\mathbf{R}$ is differentiable and $P\in Sp(2n)$ is prescribed.

Theorem 2.9. Assume that

(i) $H''(t,x)$ is continuous and is bounded for $(t,x)\in[0,1]\times\mathbf{R}^{2n}$, and $H'(t,\theta)=\theta$;

(ii) there exists B_1, $B_2\in L^\infty([0,1];GL_s(\mathbf{R}^{2n}))$ satisfying $i_P^f(B_1)=i_P^f(B_2)$, $\nu_P^f(B_2)=0$ and

$$B_1(t)\le H''(t,x)\le B_2(t)\ \ \forall(t,x)\in[0,1]\times\mathbf{R}^{2n}\ \ with\ \ |x|\ge r>0;$$

(iii)with $B_0:=H''(\cdot,\theta)$ we have

$$i_P^f(B_1)\notin[i_P^f(B_0),i_P^f(B_0)+\nu_P^f(B_0)].$$

Then (2.4)(2.10) has a nontrivial solution.

Under the further assumption that

(iii) $\nu_P^f(B_0)=0$ and $|i_P^f(B_1)-i_P^f(B_0)|\ge 2n$, (2.4)(2.9) has two nontrivial solutions.

Similar to Lemma 2.4 $\Phi\in C^2(V)$. Also similar to Theorem 2.6, Theorem 2.9 follows from Theorem 1.1.

Set $i^f(B)=i_{I_{2n}}^f(B)$ for any $B\in L^\infty([0,1];\mathrm{GL}_s(\mathbf{R}^{2n})), i^s(B)=i^f(\{B,I_n\})$ for any $B\in L^\infty([0,1];\mathrm{GL}_s(\mathbf{R}^n))$. Concerning periodic solutions of Hamiltonian systems we have the following theorems.

Theorem 2.10. Assume that

(i) $H''(t,x)$ is continuous and is bounded for $(t,x)\in[0,1]\times\mathbf{R}^{2n}$, and $H'(t,\theta)=\theta$;

(ii) there exist B_1, $B_2 \in L^\infty([0,1]; GL_s(\mathbf{R}^{2n}))$ satisfying $i^f(B_1) = i^f(B_2)$, $\nu^f(B_2) = 0$ and

$$B_1(t) \leq H''(t,x) \leq B_2(t) \quad \forall (t,x) \in [0,1] \times \mathbf{R}^{2n} \quad with \quad |x| \geq r > 0;$$

(iii)with $B_0 := H''(\cdot, \theta)$ we have

$$i^f(B_1) \notin [i^f(B_0), i^f(B_0) + \nu^f(B_0)].$$

Then (2.4) has a nontrivial periodic solution $x = x_0$.

Under the further assumption that

(iv) $\nu^f(B_0) = 0$ and $|i^f(B_1) - i^f(B_0)| \geq 2n$, (2.4) has two nontrivial periodic solutions.

Theorem 2.11. Assume that

(i) $V''(t,x)$ is continuous and is bounded for $(t,x) \in [0,1] \times \mathbf{R}^{2n}$, $V'(t,\theta) = \theta$;

(ii) there exist B_1, $B_2 \in L^\infty([0,1]; GL_s(\mathbf{R}^{n}))$ satisfying $i^s(B_1) = i^s(B_2)$, $\nu^s(B_2) = 0$ and

$$B_1(t) \leq V''(t,x) \leq B_2(t) \quad \forall (t,x) \in [0,1] \times \mathbf{R}^{n} \quad with \quad |x| \geq r > 0;$$

(iii)with $B_0 := V''(\cdot, \theta)$ we have

$$i^s(B_1) \notin [i^s(B_0), i^s(B_0) + \nu^s(B_0)].$$

Then (2.9) has a nontrivial periodic solution.

Under the further assumption that

(iv) $\nu^s(B_0) = 0$ and $|i^s(B_1) - i^s(B_0)| \geq 2n$, (2.9) has two nontrivial periodic solutions.

Remark. Theorems 2.6 and 2.9 were obtained already in [1] as special cases of a result concerning the first kind operator equation. However, the inequality (2.1) in [1] should be replaced by

$$a(x,x) + \lambda_0 ||x||_X^2 \geq c||x||_Z, \quad \forall \quad x \in Z$$

for some positive constants λ_0 and c.

3 A new reduced functional

In this section we will construct a new functional to investigate (1.1). The method comes from Section 2.1 of Chapter IV in [4] and [12]. Because every eigenvalue of A is isolated, there exists $\epsilon > 0$ such that $A_\epsilon := A + \epsilon I : D(A) \subset X \to X$ is invertible and the inverse $A_\epsilon^{-1} : X \to X$ satisfies

$$||A_\epsilon^{-1}|| \leq \frac{1}{\epsilon}. \tag{3.1}$$

Set $\Phi_\epsilon(x) = \Phi(x) + \frac{1}{2}\epsilon||x||^2$. Then (1.1) is equivalent to the following equation

$$A_\epsilon x - \Phi_\epsilon'(x) = \theta. \tag{3.2}$$

Obviously $D(A_\epsilon) = D(A), A_\epsilon : D(A) \subset X \to X$ is selfadjoint and $\sigma(A_\epsilon) = \sigma_d(A_\epsilon)$. Let $\{E_\lambda'\}$ be the spectral resolution of A_ϵ. There is an orthogonal decomposition:

$$X = X^+ \oplus X^0 \oplus X^-$$

where $X^* = P^*X$ for $* = +, 0, -$ and $P^+ = \int_0^\infty dE_\lambda'$, $P^0 = \int_{-\beta}^0 dE_\lambda'$, $P^- = \int_{-\infty}^{-\beta} dE_\lambda'$ and $\beta > 0$. And from now on we always assume that $-\beta \in \rho(A_\epsilon)$. Let $x \in D(A_\epsilon)$ be a solution of (3.2). Set $x = x^+ + x^0 + x^-, u = u^+ + u^0 + u^-, u^\pm = |A_\epsilon|^{\frac{1}{2}} x^\pm, u^0 = |A_\epsilon|^{\frac{1}{2}} x^0$. Because $A_\epsilon x = |A_\epsilon|(x^+ - x^0 - x^-)$, $u = |A_\epsilon|^{\frac{1}{2}} x$ satisfies the following equation

$$u^+ - u^0 - u^- - |A_\epsilon|^{-\frac{1}{2}} \Phi_\epsilon'(|A_\epsilon|^{-\frac{1}{2}} u) = \theta. \tag{3.3}$$

Note that $V = D(|A|^{\frac{1}{2}}) = D(|A_\epsilon|^{\frac{1}{2}})$. Similar to page 189 in [4] by Chang, we define the functional as follows

$$\varphi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^0\|^2 - \frac{1}{2}\|u^-\|^2 - \Phi_\epsilon(|A_\epsilon|^{-\frac{1}{2}} u), \quad \forall \ \ u \in X. \tag{3.4}$$

The Euler equation of this functional is (3.3). We only discuss the case:

$$\sigma(A) \ \text{ is unbounded from below.} \tag{3.5}$$

After that we will find that if $\sigma(A)$ is bounded from below, the things related are much simper. Since (3.5) holds, $\dim P^- X = +\infty$ and the Morse(negative) index at any critical point is always infinite. In order to use Morse theory to investigate (1.1) we need to obtain a reduced functional having a finite Morse index at any critical point. To this end, we use the method from [12]. Note that (3.3) is equivalent to the following system:

$$u^+ - u^0 - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}} \Phi_\epsilon'(|A_\epsilon|^{-\frac{1}{2}} u) = \theta \tag{3.6}$$

$$-u^- - P^-|A_\epsilon|^{-\frac{1}{2}} \Phi_\epsilon'(|A_\epsilon|^{-\frac{1}{2}} u) = \theta \tag{3.7}$$

We will solve (3.7) for u^0, u^+ fixed. Denote the Frechet derivative of $\Phi(x)$ with respect to x in V by $\Phi_V'(x)$. Because $(\Phi_V'(u), v)_V = (\Phi'(u), v)$ for any $u, v \in V$, (3.7) has an equivalent form

$$u^- = -P^-(|A_\epsilon|^{-\frac{1}{2}})^* \Phi_{\epsilon V}'(|A_\epsilon|^{-\frac{1}{2}} u). \tag{3.8}$$

Set $\mathcal{N}(u^-) = -P^-(|A_\epsilon|^{-\frac{1}{2}})^* \Phi_{\epsilon V}'(|A_\epsilon|^{-\frac{1}{2}}(u^+ + u^0 + u^-))$ for any $u^- \in X^-$. It suffices to prove that $\|\mathcal{N}(u_1^-) - \mathcal{N}(u_2^-)\| \le \alpha \|u_1^- - u_2^-\|$ for some fixed $\alpha \in (0, 1)$ and any $u_1^-, u_2^- \in X^-$. In fact, let

$\Phi''_V(x)$ denote the second Frechet derivative of $\Phi(x)$ with respect to x in V. It is also easy to check that $(\Phi''_V(x)u,v)_V = (\Phi''(x)u,v)$ for any $x,u,v \in V$. Thus

$$\begin{aligned}
&\mathcal{N}(u_2^-) - \mathcal{N}(u_1^-) \\
&= P^-(|A_\epsilon|^{-\frac{1}{2}})^*\Phi'_{\epsilon V}(|A_\epsilon|^{-\frac{1}{2}}(u^+ + u^0 + u_1^-)) - P^-(|A_\epsilon|^{-\frac{1}{2}})^*\Phi'_{\epsilon V}(|A_\epsilon|^{-\frac{1}{2}}(u^+ + u^0 + u_2^-)) \\
&= P^-(|A_\epsilon|^{-\frac{1}{2}})^* \int_0^1 \Phi''_{\epsilon V}(|A_\epsilon|^{-\frac{1}{2}}(u^+ + u^0 + \theta u_1^- + (1-\theta)u_2^-))d\theta |A_\epsilon|^{-\frac{1}{2}}(u_1^- - u_2^-) \\
&= P^-|A_\epsilon|^{-\frac{1}{2}} \int_0^1 \Phi''_{\epsilon}(|A_\epsilon|^{-\frac{1}{2}}(u^+ + u^0 + \theta u_1^- + (1-\theta)u_2^-))d\theta |A_\epsilon|^{-\frac{1}{2}}(u_1^- - u_2^-)
\end{aligned}$$

Let $\{e_j\}$ be the orthonormal basis of X as in Section 1 and $A_\epsilon e_j = \lambda_j^{'} e_j$ where $\lambda_j^{'} = \lambda_j + \epsilon$ are all eigenvalues of A_ϵ satisfying $\lambda_j^{'} \le \lambda_{j+1}^{'}$ $\forall j$ and $\lambda_j^{'} \to \pm\infty$ as $j \to \pm\infty$. Then for any $x = \sum c_j e_j \in X$ we have $P^-(|A_\epsilon|^{-\frac{1}{2}})x = \Sigma_{\lambda'_{j<-\beta}} c_j(-\lambda_j^{'})^{-\frac{1}{2}} e_j$ and

$$\|P^-|A_\epsilon|^{\frac{1}{2}}x\| = (\Sigma_{\lambda_j^{'}<-\beta} c_j^2(-\lambda_j^{'})^{-1})^{\frac{1}{2}} \le (\frac{1}{\beta}\Sigma c_j^2)^{\frac{1}{2}} = \frac{1}{\sqrt{\beta}}\|x\|.$$

Thus

$$\|P^-|A_\epsilon|^{-\frac{1}{2}}\| \le \frac{1}{\sqrt{\beta}}. \tag{3.9}$$

And by assumption(i) there exists $M > 0$ such that

$$||\Phi''(x)|| \le M, \quad ||\Phi''_\epsilon(x)|| \le M \quad \forall \quad x \in X. \tag{3.10}$$

Hence $\|\mathcal{N}(u_2^-) - \mathcal{N}(u_1^-)\| \le \frac{M}{\beta}\|u_2^- - u_1^-\|$. Let $\beta > 0$ be large enough such that $\frac{M}{\beta} < 1$. Then $\mathcal{N}(u^-) = u^-$ and equivalently (3.7) has a unique solution $u^- = u^-(u^+, u^0) \in C^1(X^+ \oplus X^0, X^-)$.

Define

$$a(u^+ + u^0) = \varphi(u^+ + u^0 + u^-(u^+, u^0)). \tag{3.11}$$

A critical point of $a(u^+ + u^0)$ corresponds to a solution of (3.2). In fact,

$$\begin{aligned}
a'(u^+ + u^0) \quad &= u^+ - u^0 - (u^{-'})^* u^- - (P^+ + P^0 + (u^{-'})^* P^-)(|A_\epsilon|^{-\frac{1}{2}})^*\Phi'_{\epsilon V}(|A_\epsilon|^{-\frac{1}{2}}u) \\
&= u^+ - u^0 - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}\Phi'_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u),
\end{aligned} \tag{3.12}$$

where $u = u^+ + u^0 + u^-$ and u^- satisfies (3.7). Hence, $a'(u^+ + u^0) = 0$ if and only if (3.6-3.7) hold and equivalently (3.3) holds. Thus, we have the following proposition.

Proposition 3.1 Under assumptions (i-ii) of Theorem 1.1 the functional $a(u^+ + u^0)$ defined in (3.11) belongs to $C^2(E)$, and every critical point $u^+ + u^0$ corresponds to a solution $x = |A_\epsilon|^{\frac{1}{2}}(u^+ + u^0 + u^-(u^+ + u^0))$ of (3.2).

In order to prove Theorem 1.1 we need to investigate the Morse index of $a(u^+ + u^0)$ at a critical point. Let us calculate $a''(u^+ + u^0)$ now. From (3.8) it follows that

$$- P^-|A_\epsilon|^{-\frac{1}{2}}\Phi''_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}(P^+ + P^0) = (P^- + P^-|A_\epsilon|^{-\frac{1}{2}}\Phi''_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}P^-)u^{-'}(u^*).$$

From (3.9-3.10) for $\beta > M$ the operator on the right side is invertible and

$$\begin{aligned} u^{-'}(u^*) = \quad & -(P^- + P^-|A_\epsilon|^{-\frac{1}{2}}\Phi''_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}P^-)^{-1} \\ & P^-|A_\epsilon|^{-\frac{1}{2}}\Phi''(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}(P^+ + P^0). \end{aligned}$$

Thus, (3.12) implies

$$\begin{aligned} a''(u^+ + u^0) \quad & = P^+ - P^0 - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}\Phi''_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}(P^+ + P^0 + u^{-'}(u^*)) \\ & = P^+ - P^0 - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}\Phi''_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}(P_+ + P_0) \\ & +(P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}\Phi''_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}(P^- \\ & \quad +P^-|A_\epsilon|^{-\frac{1}{2}}\Phi''_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}P^-)^{-1}P^-|A_\epsilon|^{-\frac{1}{2}}\Phi''_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}(P^+ + P^0), \end{aligned} \tag{3.13}$$

where $u = u^+ + u^0 + u^-(u^+, u^0)$.

In order to prove Theorem 1.1 we also need a lemma. Let X be a Hilbert space and $f \in C^2(X, \mathbf{R})$. As in [4, Chapter 1] let $K = \{x \in X | f'(x) = \theta\}, f_a = \{x \in X | f(x) \leq a\}$. If $f'(x) = \theta, c = f(x)$ we say that x is a critical point of f, and c is a critical value. $c \in \mathbf{R}$ is called a regular value of f if it is not a critical value. For any $x \in K$, $f''(x) \in L_s(X)$ is selfadjoint. We call the dimension of the negative subspace denoted by $m^-(f''(x))$ corresponding the the spectral decomposing the Morse index of x, and $m^0(f''(x)) := \dim ker f''(x)$ is called the Morse nullity of x. If $f''(x)$ has a bounded inverse then x is called non-degenerate. For any two topological spaces $Y \subset X$ let $H_q(X, Y; \mathbf{R})$ denote the qth regular relative homology group. For an isolated critical point x, the qth critical group is defined by $C_q(f, x) = H_q(f_c \cap U, (f_c \setminus \{x\}) \cap U); \mathbf{R})$ for any neighborhood U of x with $U \cap K = \{x\}$ and $c = f(x_0)$. From [4, Chapter II Theorems 5.1 and 5.2] we have the following lemma.

Lemma 3.2. Assume $f \in C^2(X, \mathbf{R})$ satisfies the (PS) condition, $f'(\theta) = \theta$, and there is a positive integer γ such that $\gamma \notin [m^-(f''(\theta)), m^0(f''(\theta)) + m^-(f''(\theta))]$ and $H_q(X, f_a; \mathbf{R}) = \delta_{q\gamma}\mathbf{R}$ for some regular value $a < f(\theta)$. Then f has a critical point $p_0 \neq \theta$ with $C_\gamma(f, p_0) \neq 0$. Moreover, if θ is a non-degenerate critical point, and $m^0(f''(p_0)) \leq |\gamma - m^-(f''(\theta))|$, then f has another critical point $p_1 \neq p_0, \theta$.

4 Index theory for linear self-adjoint operator equations

In this section we investigate the Morse index of $a''(u^*)$ obtained in the last section. This is a continuation to prepare for the proof of Theorem 1.1. The method comes from [13-14]. For any $B \in L_s(X)$, we set $B_\epsilon = B + \epsilon I$ where $\epsilon > 0$ satisfies (3.1), and there exist large numbers $\beta > M > 0$ such that

$$\|B\| \leq M, \|B_\epsilon\| \leq M. \tag{4.1}$$

Then

$$\|P^-|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-\| \leq \frac{M}{\beta} \tag{4.2}$$

and $P^- + P^-|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^- : X^- \to X^-$ is invertible. Motivated by (3.13) we consider the following bilinear form

$$\begin{aligned} q_{A,\beta;B}(u^*, v^*) &= \frac{1}{2}(u^+, v^+) - \frac{1}{2}(u^0, v^0) - \frac{1}{2}(|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}u^*, v^*) \\ &+ \frac{1}{2}(|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-(P^- + P^-|A|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-)^{-1}P^-|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}u^*, v^*), \end{aligned} \tag{4.3}$$

where $u^* = u^+ + u^0, v^* = v^+ + v^0$ belong to $E := X^+ \bigoplus X^0$. Define

$$\begin{aligned} \mathcal{B}u^* &= 2P^0u^* + (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}u^* \\ &-(P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}(P^- + P^-|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-)^{-1}P^-|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}u^*. \end{aligned} \tag{4.4}$$

Then $q_{A,\beta;B}(v^*, u^*) = \frac{1}{2}[(v^*, u^*) - (\mathcal{B}v^*, u^*)]$ for any $v^*, u^* \in E$. And $\mathcal{B} : E \to E$ is self-adjoint and compact. By the spectral theory there is a basis $\{e_j\}$ of E and a sequence $\mu_j \to 0$ in $\mathbf{R}$ such that:

$$\mathcal{B}e_j = \mu_j e_j; \ (e_j, e_i) = \delta_{ij}. \tag{4.5}$$

For any $u^* \in E$, which can be expressed as $u^* = \sum_{j=1}^{\infty} c_j e_j$

$$q_{A,\beta;B}(u^*, u^*) = \frac{1}{2}\sum_{j=1}^{\infty}(1 - \mu_j)c_j^2.$$

Define

$$\begin{aligned} E^-_{A,\beta}(B) : &= \{\sum_{j=1}^{\infty} c_j e_j \mid c_j = 0 \text{ if } 1 - \mu_j \geq 0\}, \\ E^0_{A,\beta}(B) : &= \{\sum_{j=1}^{\infty} c_j e_j \mid c_j = 0 \text{ if } 1 - \mu_j \neq 0\}, \\ E^+_{A,\beta}(B) : &= \{\sum_{j=1}^{\infty} c_j e_j \mid c_j = 0 \text{ if } 1 - \mu_j \leq 0\}. \end{aligned}$$

Obviously, $E^-_{A,\beta}(B)$, $E^0_{A,\beta}(B)$ and $E^+_{A,\beta}(B)$ are $q_{A,\beta;B}$-orthogonal and $E^-_{A,\beta}(B)\oplus E^0_{A,\beta}(B)\oplus E^+_{A,\beta}(B) = E$. Since $\mu_j \to 0$ as $j \to \infty$, $E^-_{A,\beta}(B)$ and $E^0_{A,\beta}(B)$ are two finite dimensional subspaces.

Definition 4.1. We define

$$\nu_{A,\beta}(B) := \dim\, E^0_{A,\beta}(B), i_{A,\beta}(B) := \dim\, E^-_{A,\beta}(B).$$

Proposition 4.2. We have the following results:

(i)$\nu_{A,\beta}(B) = dim(\ker(A - B))$;

(ii)$i_{A,\beta}(B)$ is the Morse index of $q_{A,\beta;B}$;

(iii) For any $B_0, B_1 \in L_s(X)$ satisfying $B_0 \leq B_1$ and $B_0 < B_1$ with respect to $ker(A - B_\lambda)$ $\forall\ \lambda \in [0, 1)$ where $B_\lambda = B_0 + \lambda(B_1 - B_0)$ if the subspace is not trivial, we have

$$i_{A,\beta}(B_1) - i_{A,\beta}(B_0) = \Sigma_{\lambda\in[0,1)}\nu_{A,\beta}(B_\lambda).$$

Proof. (i) Fix any $u^* \in E^0_{A,\beta}(B)$; by definition $q_{A,\beta;B}(u^*, v^*) = 0\ \ \forall\ v^* \in E$. It follows

$$(P^+ - P^0)u^* - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}u^* - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}u^- = 0, \tag{4.6}$$

where

$$u^- = -(P^- + P^-|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-)^{-1}P^-|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}u^*,$$

from which it follows

$$u^- = -P^-|A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}(u^* + u^-).$$

Let $x^- = |A_\epsilon|^{-\frac{1}{2}}u^-, x^+ + x^0 = |A_\epsilon|^{-\frac{1}{2}}u^*$, and $x = x^+ + x^0 + x^-$. We obtain

$$A_\epsilon x^- - P^-B_\epsilon x = 0. \tag{4.7}$$

From (4.6) we obtain

$$(P^+ - P^0)u^* - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u^* + |A_\epsilon|^{-\frac{1}{2}}u^-) = 0.$$

That is

$$A_\epsilon(x^+ + x^0) - (P^+ + P^0)B_\epsilon x = 0. \tag{4.8}$$

Combining (4.7) and (4.8) implies

$$Ax - Bx = 0.$$

Thus, from the one to one correspondence $u^* \mapsto x = |A_\epsilon|^{-\frac{1}{2}}(u^* + u^-)$ it follows that $E^0_{A,\beta}(B) \cong \ker(A - B)$.

(ii) Assume X_1 is a subspace of E such that $q_{A,\beta;B}$ is negative definite on X_1 with $\dim X_1 = k$. Let $\{x_j\}_1^k$ be linear independent in X_1. We have the decomposition $x_j = x_j^- + x_j^*$ with $x_j^- \in E^-_{A,\beta}(B)$ and $x_j^* \in E^0_{A,\beta}(B) \oplus E^+_{A,\beta}(B)$. If there exist not all zero numbers $\alpha_i \in \mathbf{R}$ such that $\sum_{i=1}^k \alpha_i x_i^- = \theta$. On the one hand, $x := \sum_{i=1}^k \alpha_i x_i \in X_1 \setminus \{\theta\}$ and $q_{A,\beta;B}(x,x) < 0$; on the other hand, $x = \sum_{i=1}^k \alpha_i x_i^* \in E^0_{A,\beta}(B) \oplus E^+_{A,\beta}(B)$, and $q_{A,\beta;B}(x,x) \geq 0$. So $\{x_i^-\}_{i=1}^k$ is linear independent and $i_{A,\beta}(B) \geq k = \dim X_1$.

(iii) From Theorem 1.5(ii) and (1.4), $i_A(B_1) - i_A(B_0) = \Sigma_{\lambda\in[0,1)}\nu_A(B_\lambda)$, and there are at most finite numbers $\lambda \in [0,1)$ such that $ker(A - B_\lambda) \neq \{\theta\}$. Set $i(\lambda) = i_{A,\beta}(B_\lambda)$, $\nu(\lambda) = \nu_{A,\beta}(B_\lambda)$ for $\lambda \in [0,1)$. Let $\mathcal{B}(\lambda)$ be the operator similar to $\mathcal{B}$ in (4.4) only with B_ϵ replaced with $B_\lambda + \epsilon I$. We have the following lemma.

Lemma 4.3. Let $M > 0$ be large enough such that $\|B_0\| \leq M$, $\|B_1\| \leq M$, $\epsilon < M$ and $4M + \epsilon < \beta$. Then we have the following expression

$$\mathcal{B}(\lambda) = \mathcal{B}(\lambda_0) + (\lambda - \lambda_0)\mathcal{B}_1(\lambda_0) + \cdots + (\lambda - \lambda_0)^k \mathcal{B}_k(\lambda_0) + \cdots,$$

where $\mathcal{B}_k : E \to E$ is selfadjoint and compact and satisfies

$$\|\mathcal{B}_k(\lambda_0)\| \leq \frac{8M}{\epsilon}, \quad k = 2, 3, 4 \cdots,$$

and

$$\mathcal{B}_1(\lambda_0) = (P^+ + P^0)(I + S^T)|A_\epsilon|^{-\frac{1}{2}}(B_1 - B_0)|A_\epsilon|^{-\frac{1}{2}}(I + S)(P^+ + P^0),$$
$$S = -P^-(P^- + P^-|A_\epsilon|^{-\frac{1}{2}}B_{\lambda_0}|A_\epsilon|^{-\frac{1}{2}}P^-)^{-1}P^-|A_\epsilon|^{-\frac{1}{2}}B_{\lambda_0}|A_\epsilon|^{-\frac{1}{2}}.$$

Proof. We have

$$\begin{aligned}
&P^- + P^-|A_\epsilon|^{-\frac{1}{2}}(B_\lambda)_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^- \\
&= (P^- + P^-|A_\epsilon|^{-\frac{1}{2}}(B_{\lambda_0})_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-)(P^- \\
&\quad + (\lambda - \lambda_0)(P^- + P^-|A_\epsilon|^{-\frac{1}{2}}(B_{\lambda_0})_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-)^{-1}P^-|A_\epsilon|^{-\frac{1}{2}}(B_1 - B_0)|A_\epsilon|^{-\frac{1}{2}}P^-). \\
&:= Q_1(P^- + (\lambda - \lambda_0)Q_1^{-1}Q_2),
\end{aligned}$$

where $Q_1 = P^- + P^-|A_\epsilon|^{-\frac{1}{2}}(B_{\lambda_0})_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-$, $Q_2 = P^-|A_\epsilon|^{-\frac{1}{2}}(B_1 - B_0)|A_\epsilon|^{-\frac{1}{2}}P^-$, $\|Q_1^{-1}\| \leq \frac{1}{1 - \frac{M+\epsilon}{\beta}}$, $\|Q_2\| \leq \frac{2M}{\beta}$.

$$(P^- + P^-|A_\epsilon|^{-\frac{1}{2}}(B_\lambda)_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-)^{-1}$$

$$= (P^- + (\lambda - \lambda_0)Q_1^{-1}Q_2)^{-1}Q_1^{-1}$$
$$= Q_1^{-1} - (\lambda - \lambda_0)Q_1^{-1}Q_2Q_1^{-1} + (\lambda - \lambda_0)^2(Q_1^{-1}Q_2)^2Q_1^{-1} + \cdots$$

Thus, the third term in $\mathcal{B}(\lambda)$ is

$$\begin{aligned}&-Q_3Q_1^{-1}Q_3^T + (\lambda - \lambda_0)[Q_3Q_1^{-1}Q_2Q_1^{-1}Q_3^T - Q_3Q_1^{-1}Q_4^T - Q_4Q_1^{-1}Q_3^T] + \cdots \\ &+(\lambda - \lambda_0)^k[(-1)^{k+1}Q_3(Q_1^{-1}Q_2)^kQ_1^{-1}Q_3^T + (-1)^kQ_4(Q_1^{-1}Q_2)^{k-1}Q_1^{-1}Q_3^T \\ &+(-1)^kQ_3(Q_1^{-1}Q_2)^{k-1}Q_1^{-1}Q_4^T + (-1)^{k-1}Q_4(Q_1^{-1}Q_2)^{k-2}Q_1^{-1}Q_4^T] + \cdots,\end{aligned}$$

where $Q_3 = (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}(B_{\lambda_0})_\epsilon|A_\epsilon|^{-\frac{1}{2}}P^-$, $Q_4 = (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}(B_1 - B_0)|A_\epsilon|^{-\frac{1}{2}}P^-$. Then $\|Q_3\| \leq \frac{2M}{\sqrt{\epsilon\beta}}$, $\|Q_4\| \leq \frac{2M}{\sqrt{\epsilon\beta}}$, $\|Q_1^{-1}\| \leq \frac{1}{1-\frac{M+\epsilon}{\beta}}$ and $\|Q_2\| \leq \frac{2M}{\beta}$. Thus, the results follow.

Now we give the proof in three steps.

Step 1. If $\nu(\lambda) = 0$, then $i(\lambda)$ is continuous. Suppose that $\nu(\lambda_0) = 0$ for some $\lambda_0 \in (0,1)$. Let S_1 be the unit ball of $E^-_{A,\beta}(B_{\lambda_0})$. Because the subspace is finite dimensional, S_1 is compact and $q_{A,\beta;B_\lambda}(u,u)$ is continuous with respect to $(\lambda, u) \in [0,1] \times S_1$. Thus, for λ close enough to λ_0, $q_{A,\beta;B_\lambda}$ is negative definite on S_1 and hence on $E^-_{A,\beta}(B_{\lambda_0})$. By (ii), $i(\lambda) \geq i(\lambda_0)$. In the following we prove the inverse inequality. If $i(\lambda_l) > i(\lambda_0) := k$ for $\lambda_l \to \lambda_0$, then similar to (4.5) we have

$$\mathcal{B}(\lambda_l)e_{l,j} = \mu_{l,j}e_{l,j};\ (e_{l,j}, e_{l,i}) = \delta_{ij} \text{ for any } 1 \leq i,j \leq k+1, \tag{4.9}$$

where $\text{span}\{e_{l,j}\} \subseteq E^-_{A,\beta}(B_{\lambda_l})$. By definition $\mu_{l,j} = (\mathcal{B}(\lambda_l)e_{l,j}, e_{l,j})$ is bounded in $\mathbf{R}$ for $j = 1, \cdots k+1$, and $\lambda_l \in [0,1]$. So we can assume $e_{l,j} \rightharpoonup e_j$, $\mu_{l,j} \to \mu_j$ and $\mathcal{B}(\lambda_l)e_{l,j} \to \mathcal{B}(\lambda_0)e_j$ by gong to subsequences if necessary. Taking the limit in (4.9) we obtain

$$\mathcal{B}(\lambda_0)e_j = \mu_je_j. \tag{4.10}$$

By definition for $j = 1, \cdots k+1$, $1 + \mu_{l,j} < 0$ and $\{\frac{1}{\mu_{l,j}}\}$ is bounded in $\mathbf{R}$. So

$$e_{l,j} = \frac{1}{\mu_{l,j}}\mathcal{B}(\lambda)e_{l,j} \to \frac{1}{\mu_j}\mathcal{B}(\lambda_0)e_j = e_j$$

in E, and $(e_i, e_j) = \delta_{ij}$ $1 \leq i,j \leq k+1$. It follows that $\{e_j\}_1^{k+1}$ is independent. And for every $u = \sum_{i=1}^{k+1} c_je_j$, since $\sum_{i=1}^{k+1} c_je_{l,j} \to u$ in E and

$$q_{A,\beta;B_{\lambda_l}}(\sum_{i=1}^{k+1} c_je_{l,j}, \sum_{i=1}^{k+1} c_je_{l,j}) < 0,$$

taking the limit as $l \to \infty$ we have

$$q_{A,\beta;B_{\lambda_0}}(u,u) \leq 0.$$

This means that $i(\lambda_0) \geq k+1$, a contradiction.

Step 2. If $\nu(\lambda_0) = 0$ does not hold for $\lambda_0 \in [0,1)$, then $i(\lambda_0+0) = i(\lambda_0) + \nu(\lambda_0)$. From the argument above it follows that $i(\lambda_0+0) \leq i(\lambda_0) + \nu(\lambda_0)$. Thus, by (ii) we need only to prove that $q_{A,\beta;B_\lambda}(u,u) \leq 0$ for every $u \in E^-_{A,\beta}(B_{\lambda_0}) \oplus E^0_{A,\beta}(B_{\lambda_0})\backslash\{\theta\}$ with $\lambda - \lambda_0 > 0$ small. Let $S = \{u \in E^-_{A,\beta}(B_{\lambda_0}) \oplus E^0_{A,\beta}(B_{\lambda_0}) | \|u\| = 1\}$. Because S is compact, we need only to prove that $\forall\ \ u^* \in S$ there exists $\delta > 0$ that for any $\delta > \lambda - \lambda_0 > 0$, $q_{A,\beta;B_\lambda}(v^*, v^*) < 0$ if v^* is close to u^*.

Case 1. Assume $u^* = u^- + u^0 \in S$, $u^- \neq \theta$. It follows

$$q_{A,\beta;B_\lambda}(v^*, v^*) = q_{A,\beta;B_{\lambda_0}}(v^*, v^*) + o(1)$$

as $\lambda \to \lambda_0^+$.

Because $q_{A,\beta;B_{\lambda_0}}(u^-, u^-) < 0$, there exists a neighborhood U of u^* in S such that $\forall\ \ v^* \in U$

$$q_{A,\beta;B_{\lambda_0}}(v^*, v^*) < \frac{1}{2} q_{A,\beta;B_{\lambda_0}}(u^-, u^-).$$

The results follows.

Case 2. Assume $u^* \in S \bigcap E^0_{A,\beta}(B_{\lambda_0})$.

By the proof of (i) $(\mathcal{B}_1(\lambda_0)u^*, u^*) > 0$. There exists a neighborhood U in S such that $\forall v^* \in U$, $(\mathcal{B}_1(\lambda_0)v^*, v^*) > \frac{1}{2}(\mathcal{B}_1(\lambda_0)u^*, u^*)$. It follows from Lemma 4.3 that

$$\begin{aligned} q_{A,\beta;B_\lambda}(v^*, v^*) \qquad & \leq q_{A,\beta;B_{\lambda_0}}(v^*, v^*) - \frac{1}{2}(\lambda - \lambda_0)(\mathcal{B}_1(\lambda_0)v^*, v^*) + o(\lambda - \lambda_0) \\ & \leq -\frac{1}{4}(\lambda - \lambda_0)(\mathcal{B}_1(\lambda_0)u^*, u^*) + o(\lambda - \lambda_0) \\ & < 0, \quad \text{as} \lambda \to \lambda_0^+. \end{aligned}$$

Step 3. If $\nu(\lambda_0) \neq 0$ for same $\lambda_0 \in (0,1)$, then $i(\lambda_0 - 0) = i(\lambda_0)$. In fact, suppose $i(\lambda_0 - 0) > i(\lambda_0)$. As in Step 1, there exists $\lambda_l \to \lambda_0^-$ and $\mu_l \in \sigma(\mathcal{B}(\lambda_l))$ satisfying $\mu_l \to 1^+$. However, Lemma 4.3 tells us

$$\mathcal{B}(\lambda) = \mathcal{B}(\lambda_0) + (\lambda - \lambda_0)\mathcal{B}_1(\lambda_0) + o(\lambda - \lambda_0).$$

By Rellich's theory(Theorem 3.9 and Remark 3.11 in pages 392-393 of [15]), its eigenvalue and associated eigenvector are holomorphic:

$$\begin{aligned} &\mu(\lambda) = 1 + (\lambda - \lambda_0)\mu_1 + o(\lambda - \lambda_0), \\ &u(\lambda) = u_0 + (\lambda - \lambda_0)u_1 + +o(\lambda - \lambda_0), B(\lambda_0)u_0 = u_0 \neq \theta. \end{aligned}$$

Thus, $\mu_1 = (\mathcal{B}_1(\lambda_0)u_0, u_0) > 0$ and $\mu_l = \mu(\lambda_l) = 1 + (\lambda_l - \lambda_0)\mu_1 + o(\lambda_l - \lambda_0) < 1$ as $\lambda_l \to \lambda_0^-$. This is a contradiction. ∎

Note that $q_{A,\beta;B}(u,u)$ is not monotone with respect to B because of the last term. This is not convenient to the proof of Theorem 1.1. However, the inequality (3.9) implies that the last term is much smaller than the third term if $\beta > 0$ is large enough. Thus, we can use the sum of the first three terms denoted by $\bar{q}_{A,\beta;B}(u,u)$ instead, and we have the following proposition.

Proposition 4.3 (i) Assume that $B \in L_s(X)$ satisfies $\nu_{A,\beta}(B) = 0$. Then for $\beta > 0$ large enough, $(\bar{q}_{A,\beta;B}(u,u))^{\frac{1}{2}}$ and $(-\bar{q}_{A,\beta;B}(u,u))^{\frac{1}{2}}$ are equivalent norms on $E^+_{A,\beta}(B)$ and $E^-_{A,\beta}(B)$ respectively.

(ii) Assume that $B_1, B_2 \in L_s(X)$ satisfy $B_1 < B_2, i_{A,\beta}(B_1) = i_{A,\beta}(B_2)$ and $\nu_{A,\beta}(B_1) = i_{A,\beta}(B_2) = 0$. Then, for $\beta > 0$ large enough $E = E^+_{A,\beta}(B_2) \bigoplus E^-_{A,\beta}(B_1)$.

Proof. Recall that $P^+ = \int_0^{+\infty} dE'_\lambda$, $P^0 = P^0_\beta = \int_{-\beta}^0 dE'_\lambda$, $P^- = P^-_\beta = \int_{-\infty}^{-\beta} dE'_\lambda X$ and the operator $\mathcal{B}$ defined in (4.4) depends on β, so we denote it by $\mathcal{B}_\beta$ now.

(i) We need only to prove that there exists $\delta > 0$ such that

$$(1-\delta, 1+\delta) \bigcap \sigma(\mathcal{B}_\beta) = \emptyset \tag{4.11}$$

if $\beta > \beta_0$ for some $\beta > 0$ large enough. Otherwise, there exist $\beta_k \to +\infty, \mu_k \to 1$ and unit vector $e^*_k \in E$ satisfying

$$\mathcal{B}_{\beta_k} e^*_k = \mu_k e^*_k.$$

By definition we have

$$(\mu_k - 2)P^0_k e^*_k + \mu_k P^+ e^*_k = (P^+ + P^0_k)|A_\epsilon|^{-\frac{1}{2}} B_\epsilon |A_\epsilon|^{-\frac{1}{2}} e^*_k - (P^+ + P^0_k)|A_\epsilon|^{-\frac{1}{2}} B_\epsilon |A_\epsilon|^{-\frac{1}{2}} e^-_k, \tag{4.12}$$

where $e^-_k = (P^-_k + P^-_k |A_\epsilon|^{-\frac{1}{2}} B_\epsilon |A_\epsilon|^{-\frac{1}{2}} P^-_k)^{-1} P^-_k |A_\epsilon|^{-\frac{1}{2}} B_\epsilon |A_\epsilon|^{-\frac{1}{2}} e^*_k$, $P^0_k = P^0_{\beta_k}$, $P^-_k = P^-_{\beta_k}$ and $\mathcal{B}_k = \mathcal{B}_{\beta_k}$. Assume $e^*_k \rightharpoonup e = e^+ + e^0$, then $e^+_k \rightharpoonup e^+$, $e^0_k \rightharpoonup e^0$ and $|A_\epsilon|^{-\frac{1}{2}} e^*_k \to |A_\epsilon|^{-\frac{1}{2}} e$. By (4.11), $e^+_k \to e^+ = P^+ |A_\epsilon|^{-\frac{1}{2}} B_\epsilon |A_\epsilon|^{-\frac{1}{2}} e$, $e^0_k \to e^0 = (I - P^+)|A_\epsilon|^{-\frac{1}{2}} B_\epsilon |A_\epsilon|^{-\frac{1}{2}} e$. Thus $e^*_k \to e$ and $\|e\| = \|e^*_k\| = 1$. On the other hand, it follows that $e^+ + e^0 = |A_\epsilon|^{-\frac{1}{2}} B_\epsilon |A_\epsilon|^{-\frac{1}{2}} e$, and $A_\epsilon x = B_\epsilon x$ with $x = |A_\epsilon|^{-\frac{1}{2}} e \neq 0$, a contradiction.

By definition, for $\beta \geq \beta_0$ and $u = \sum_{1-\mu_j>0} c_j e_j \in E^+_{A,\beta}(B)$ it follows

$$\sup\{1 - \mu_j | 1 - \mu_j > 0\} ||u||^2 \geq q_{A,\beta;B}(u,u)) \geq \delta ||u||^2. \tag{4.13}$$

And by (4.1), (4.2), (4.3),

$$|q_{A,\beta;B}(u,u) - \bar{q}_{A,\beta;B}(u,u)| \leq \frac{M^2}{\epsilon(\beta - M)} \|u\|^2. \tag{4.14}$$

Thus, for $\beta > 0$ large enough, $(\bar{q}_{A,\beta;B}(u,u))^{\frac{1}{2}}$ is an equivalent norm in $E^+_{A,\beta}(B)$.

(ii) Assume u belongs both $E^+_{A,\beta}(B_2)$ and $E^-_{A,\beta}(B_1)$. By definition and the result in (i),

$$0 \geq \bar{q}_{A,\beta;B_1}(u,u) \geq \bar{q}_{A,\beta;B_2}(u,u) \geq 0.$$

Thus, $u = \theta$. Now we need only to prove that $E = E^+_{A,\beta}(B_2) + E^-_{A,\beta}(B_1)$. In fact, let $\{e_j\}_1^k$ be a basis of $E^-_{A,\beta}(B_1)$. We have the decomposition $e_j = e_j^- + e_j^+$ with $e_j^- \in E^-_{A,\beta}(B_2)$ and $e_j^+ \in E^+_{A,\beta}(B_2)$. If there exist not all zero numbers $c_j \in \mathbf{R}$ such that $\sum_{j=1}^k c_j e_j^- = \theta$. On the one hand, $x := \sum_{i=1}^k c_j e_j \in E^-_{A,\beta}(B_1) \setminus \{\theta\}$ and $\bar{q}_{A,\beta;B_1}(x,x) < 0$ if $\beta > 0$ is large enough; on the other hand, $x = \sum_{j=1}^k c_j e_j^+ \in E^+_{A,\beta}(B_2)$, and $\bar{q}_{A,\beta;B_2}(x,x) \geq 0$ if $\beta > 0$ is large enough. This is a contradiction. So $\{e_j^-\}_{j=1}^k$ is linear independent. For any $u \in E$, $u = u^- + u^+$ with $u^- \in E^-_{A,\beta}(B_2), u^+ \in E^+_{A,\beta}(B_2)$. There exist $\{c_j\}_{j=1}^k \subset \psi\mathbf{R}$ such that $u^- = \sum_{j=1}^k c_j e_j^-$. Thus, $u = \sum_{j=1}^k c_j e_j + (u^+ - \sum_{j=1}^k c_j e_j^+)$. ∎

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We have the following two propositions.

Proposition 5.1. Under assumptions (i-ii) of Theorem 1.1 $a(u^*)$ satisfies the (PS) condition.

Proposition 5.2. For $c > 0$(such that $-c < f(0)$) large enough, we have

$$H_q(E; a_{-c}, \mathbf{R}) \cong \delta_{q\gamma}\mathbf{R},\ q = 0, 1, 2, \cdots, \tag{5.1}$$

where $\gamma = i_{A,\beta}(B_1)$.

Proof of Proposition 5.1. Assume that $\{u_n^+ + u_n^0\}$ is a sequence in E such that $a'(u_n^+ + u_n^0) \to 0$ in E. From (3.12),

$$\begin{aligned} a'(u_n^+ + u_n^0) &= u_n^+ - u_n^0 - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}\Phi'_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u_n) \\ u_n^- &= -P^-|A_\epsilon|^{-\frac{1}{2}}\Phi'_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u_n), \quad u_n = u_n^+ + u_n^0 + u_n^-. \end{aligned}$$

We claim that $\{u_n^+ + u_n^0\}$ is bounded. If the case is not true, then $||u_n|| \geq \|u_n^+ + u_n^0\| \to \infty$. From assumption (ii) it follows $\Phi'_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u_n) = B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u_n)|A_\epsilon|^{-\frac{1}{2}}u_n + C_n$ satisfying $C_n \in X$ is bounded and

$$B_1 \leq B(|A_\epsilon|^{-\frac{1}{2}}u_n) \leq B_2.$$

Let $y_n = u_n/\|u_n\|$. Then

$$y_n^+ - y_n^0 - y_n^- - |A_\epsilon|^{-\frac{1}{2}}B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u_n)|A|^{-\frac{1}{2}}y_n - \|u_n\|^{-1}|A_\epsilon|^{-\frac{1}{2}}C_n \to 0. \tag{5.2}$$

Because $||y_n|| = 1$, $B(|A_\epsilon|^{-\frac{1}{2}}u_n) \in L_s(X)$ is bounded and X is separable, we can assume as in [1, page 81] that $y_n \rightharpoonup y$ in X and $B(|A_\epsilon|^{-\frac{1}{2}}u_n)x \rightharpoonup Bx$ for any $x \in X$ and some $B \in L_s(X)$ such that $B_1 \le B \le B_2$, by going to subsequence if necessary. Because $|A_\epsilon|^{-\frac{1}{2}} : E \to E$ is compact, $|A_\epsilon|^{-\frac{1}{2}}B_\epsilon(u_n)|A_\epsilon|^{-\frac{1}{2}}y_n \to |A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}y$, and from (5.2) it follows that $y_n \to y$ and

$$y^+ - y^0 - y^- - |A_\epsilon|^{-\frac{1}{2}}B_\epsilon|A_\epsilon|^{-\frac{1}{2}}y = 0$$

Set $x = |A_\epsilon|^{-\frac{1}{2}}y$. Then $x \ne 0$ since $||y|| = 1$, and $Ax - Bx = 0$. This is impossible because Proposition 1.5(iv) implies that $\nu_A(B) = 0$. And the proof is complete. ∎

In order to prove Proposition 5.2 we need the following two lemmas.

Lemma 5.3. Suppose assumptions (i-ii) in Theorem 1.1 hold. Then there exists $R_0 > 0$ such that

$$< a'(u_1 + u_2), u_2 - u_1 > 1 \text{ as } \|u_2\| \ge R_0, \text{ or } \|u_1\| \ge R_0, \tag{5.3}$$

where $u_2 \in E^+_{A,\beta}(B_2)$ and $u_1 \in E^-_{A,\beta}(B_1)$.

Proof. For any $u^+ + u^0 = u_1 + u_2$ with $u_2 \in E^+_{A,\beta}(B_2)$ and $u_1 \in E^-_{A,\beta}(B_1)$, from (3.12) and assumption (ii) we have

$$\begin{aligned}
< a'(u^+ + u^0), u_2 - u_1 >= &\quad (u^+ - u^0 - (P^+ + P^0)|A_\epsilon|^{-\frac{1}{2}}(B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}u + C(|A_\epsilon|^{-\frac{1}{2}}u)), u_2 - u_1) \\
= &\quad ((P^+ - P^0)u_2 - (P^+ + P^0)B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}u_2, u_2) \\
&\quad -((P^+ - P^0)u_1 - (P^+ + P^0)B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}u_1, u_1) + r(u_1, u_2, \beta) \\
\ge &\quad \bar{q}_{A,\beta;B_2}(u_2, u_2) - \bar{q}_{A,\beta;B_1}(u_1, u_1) + r(u_1, u_2, \beta),
\end{aligned} \tag{5.4}$$

where $u = u_2 + u_1 + u^-$ and

$$\begin{aligned}
u^- &= -P^-|A_\epsilon|^{-\frac{1}{2}}\Phi'_\epsilon(u)(|A_\epsilon|^{-\frac{1}{2}}u) = -P^-|A_\epsilon|^{-\frac{1}{2}}(B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}u + C(|A_\epsilon|^{-\frac{1}{2}}u)) \\
r(u_1, u_2, \beta) &= -((P^+ + P_0)|A_\epsilon|^{-\frac{1}{2}}(B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)|A_\epsilon|^{-\frac{1}{2}}u^- + C(|A_\epsilon|^{-\frac{1}{2}}u)), u_2 - u_1).
\end{aligned}$$

From assumption (ii), let $M > 0$ such that $\|B_\epsilon(x)\| \le M$, $\|C(x)\| \le M$ $\forall x \in X$. A simple calculation shows that

$$||u^-|| \le \frac{\sqrt{\beta}M}{(\beta - M)\sqrt{\epsilon}}\|u^*\| + \frac{M^2\sqrt{\beta}}{\beta - M}, \tag{5.5}$$

and

$$|r(u_1, u_2, \beta)| \le M(\frac{M}{(\beta - M)\epsilon}(\|u_1\| + \|u_2\|) + 1 + \frac{M\sqrt{\beta}}{\beta - M})(\|u_1\| + \|u_2\|).$$

Then (5.3) follows from Proposition 4.3(i) and (5.4). The proof is complete. ∎

Lemma 5.4. Let R_0 be defined in Lemma 5.3. Then $a(u_2+u_1)\to-\infty$ uniformly for $u_2\in E^+_{A,\beta}(B_2)\cap B_{R_0}$ as $\|u_1\|\to+\infty$.

Proof. Because

$$\begin{aligned}\Phi(|A_\epsilon|^{-\frac{1}{2}}u)&=\int_0^1(\Phi'_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u\theta),|A_\epsilon|^{-\frac{1}{2}}u)d\theta\\&=\int_0^1(B_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u\theta)|A_\epsilon|^{-\frac{1}{2}}u\theta+C(|A_\epsilon|^{-\frac{1}{2}}u\theta),|A_\epsilon|^{-\frac{1}{2}}u)d\theta\\&\geq\frac{1}{2}((B_1)_\epsilon|A_\epsilon|^{-\frac{1}{2}}u,|A_\epsilon|^{-\frac{1}{2}}u)-\frac{M}{\sqrt{\epsilon}}||u||.\end{aligned}\tag{5.6}$$

By assumption (ii), for any $u^0+u^+=u_1+u_2$ with $u_2\in E^+_{A,\beta}(B_2)\cap B_{R_0}, u_1\in E^-_{A,\beta}(B_1)$,

$$\begin{aligned}a(u_2+u_1)\qquad&=\frac{1}{2}\|u^+\|^2-\frac{1}{2}\|u^0\|^2-\frac{1}{2}\|u^-\|^2-\Phi_\epsilon(|A_\epsilon|^{-\frac{1}{2}}u)\\&\leq\frac{1}{2}\bar{q}_{A,\beta;B_1}(u_1,u_1)+\frac{C_2}{\sqrt{\beta}}\|u_1\|^2+C_3\end{aligned}$$

for β larger enough and some constants $C_2>0$, $C_3>0$. Here we used (5.4) and (5.5). By (4.11) (4.13) and (4.14) $\frac{1}{4}\bar{q}_{A,\beta;B_1}(u_1,u_1)+\frac{C_2}{\sqrt{\beta}}\|u_1\|^2\leq 0$ for β larger enough. Thus, $a(u_2+u_1)\to-\infty$ as $\|u_1\|\to\infty$ uniformly for $u_2\in E^+_{A,\beta}(B_2)$ for $\beta>0$ larger enough. The proof is complete. ∎

We will use Lemma 3.2 again to investigate critical points of the functional $a(u^++u^0)$ and we need to calculate some relative homology groups. As in Chang[5] and in Mawhin-Willem[16] we say that the topological space pair (X',Y') with $X'\subset Y'$ is the deformation retract of a topological space pair (X,Y) with $Y\subset X$ if $X'\subset X$, $Y'\subset Y$ and there exists $\eta:[0,1]\times X\to X$ satisfying

$$\eta(0,\cdot)=id_X,\eta(1,X)\subset X',\eta(1,Y)\subset Y,\eta(t,Y)\subset Y,$$

and

$$\eta(t,\cdot)=id_{X'},\forall t\in[0,1].$$

It is well-known that if (X',Y') is a deformation retract of (X,Y), then

$$H_q(X,Y;\mathbf{R})\cong H_q(X',Y';\mathbf{R}),q=0,1,2,\cdots.$$

For any $X\subset Y\subset Z$ if there exists $\tau:[0,1]\times Y\to Y$ satisfying $\tau(0,\cdot)=id_Y,\tau(1,Y)\subset Z$ and $\tau(t,\cdot)_Z=id_Z$, then Z is called a strong deformation retract of Y. And from a result in [16, page 171] by Mawhin-Willem we have

$$H_q(X,Y;\mathbf{R})\cong H_q(X,Z;\mathbf{R}),q=0,1,2,\cdots.$$

Proof of Proposition 5.2 Set $\mathcal{M}_{R_0} = (E^+_{A,\beta}(B_2)\bigcap B_{R_0})\bigoplus E^-_{A,\beta}(B_1), \sigma(t,u) = e^{-t}u_2 + e^t u_1$ and $T_u = \ln||u_2|| - \ln R_0$ if $u = u_2 + u_1$ and $\|u_2\| > R_0$. Here R_0 is defined in Lemma 5.3 and $B_R := \{u \in E| \|u\| \leq R\}$. Define

$$\begin{aligned} \eta(t, u_2 + u_1) \qquad &= u_2 + u_1, ||u_2|| \leq R_0, \\ &= \sigma(T_u t, u), ||u_2|| > R_0. \end{aligned}$$

By Lemma 5.3 it is easy to verify that $(\mathcal{M}_{R_0}, \mathcal{M}_{R_0}\bigcap a_c)$ is a deformation retract of (E, a_c) for any $c \in \mathbf{R}$. And hence,

$$H_q(E, a_c; \mathbf{R}) = H_q(\mathcal{M}_{R_0}, \mathcal{M}_{R_0}\bigcap a_c; \mathbf{R}), q = 0, 1, 2, \cdots. \tag{5.7}$$

Now we begin to prove that

$$H_q(\mathcal{M}_{R_0}, \mathcal{M}_{R_0}\bigcap a_{-c}; \mathbf{R}) \cong \delta_{q\gamma}\mathbf{R},$$

where $-c < a(\theta)$. By Lemma 5.4, there exist $T > 0, c_1 > c_2 > T, R_1 > R_2 > R_0$ such that

$$\mathcal{N}_{R_1} \subset a_{-c_1}\bigcap \mathcal{M}_{R_0} \subset \mathcal{N}_{R_2} \subset a_{-c_2}\bigcap \mathcal{M}_{R_0} \subset \mathcal{N}_{R_0},$$

where $\mathcal{N}_R := (E^+_{A,\beta}(B_2)\bigcap B_{R_0})\bigoplus(E^-_{A,\beta}(B_1)\backslash B_R)$ for any $R > 0$. For any $u \in \mathcal{M}_{R_0}\bigcap(a_{-c_2} \setminus a_{-c_1})$, since $\sigma(t,u) = e^{-t}u_2 + e^t u_1, a(\sigma(t,u))$ is continuous with respect to t and $a(\sigma(0,u)) = a(u) \in (-c_1, -c_2]$. From (2.21)

$$\begin{aligned} &\frac{d}{dt}a(\sigma(t,u)) = \langle a'(\sigma(t,u)), \sigma'(t,u)\rangle \\ &= \langle a'(e^{-t}u_2 + e^t u_1), -e^{-t}u_2 + e^t u_1\rangle \leq -1 \end{aligned}$$

as $t > 0$. So the time $t = T_1(u)$ arriving at $a_{-c_1}\bigcap \mathcal{M}_{R_0}$ exists uniquely and is defined by $a(\sigma(t,u)) = -c_1$. The continuity of $t = T_1(u)$ comes from the implicit function theorem. Define

$$\begin{aligned} \eta_1(t,u) \qquad &= u, \quad x \in a_{-c_1}\bigcap \mathcal{M} \\ &= \sigma(T_1(u)t, u), \quad u \in \mathcal{M}\bigcap(a_{-c_2}\backslash a_{-c_1}); \end{aligned}$$

and

$$\begin{aligned} \eta_2(t,u) \qquad &= u, \|u_1\| \geq R_1 \\ &= u_2 + tu_1 + (1-t)\frac{u_1}{\|u_1\|}R_1, \ \|x_1\| < R_1. \end{aligned}$$

By the map $\eta(t,u)=\eta_2(t,\eta_1(t,u))$ we can verify that $(E^+_{A,\beta}(B_2)\bigcap B_{R_0})\oplus(E^-_{A,\beta}(B_1)\backslash int(B_{R_1})$ is a strong deformation retract of $\mathcal{M}\bigcap a_{-c_2}$:

$$\eta(t,u)=u\ \ \forall\ \ u\in(E^+_{A,\beta}(B_2)\bigcap B_{R_0})\oplus(E^-_{A,\beta}(B_1)\backslash int(B_{R_1}),t\in[0,1],$$
$$\eta(0,u)=u\ \ and\ \ \eta(1,u)\in(E^+_{A,\beta}(B_2)\bigcap B_{R_0})\oplus(E^-_{A,\beta}(B_1)\backslash int(B_{R_1}),\ \ \forall\ \ u\in\mathcal{M}\bigcap a_{-c_2}.$$

From a result in [16, page 171] it follows that

$$\begin{aligned}&H_q(\mathcal{M},\mathcal{M}\bigcap a_{-c_2};\mathbf{R})\\ &\cong H_q((E^+_{A,\beta}(B_2)\bigcap B_{R_0})\bigoplus E^-_{A,\beta}(B_1),(E^+_{A,\beta}(B_2)\bigcap B_{R_0})\bigoplus(E^-_{A,\beta}(B_1)\backslash int(B_{R_1}));\mathbf{R})\\ &\cong H_q(E^-_{A,\beta}(B_1)\bigcap B_{R_1},\partial(E^-_{A,\beta}(B_1)\bigcap B_{R_1});\mathbf{R})\\ &\cong\delta_{q\gamma}\mathbf{R}.\end{aligned}$$

Therefore, combining (5.7) implies that (5.1) holds and the proof is complete. ∎

Proof of Theorem 1.1. From Proposition 4.2 (iii), Definition 1.3 and Theorem 1.5(ii)

$$i_{A,\beta}(B_1)-m^-(a''(\theta))=i_{A,\beta}(B_1)-i_{A,\beta}(B_0)=i_A(B_1)-i_A(B_0).$$

So assumption (iii) implies that $\gamma=i_{A,\beta}(B_1)\notin[i_{A,\beta}(B_0),i_{A,\beta}(B_0)+\nu_{A,\beta}(B_0)]$. And $\nu_{A,\beta}(B_0)=0$ means that θ is a non-degenerate critical point; $\nu_A(\Phi''(x_0))\leq|i_A(B_1)-i_A(B_0)|$ implies that $m^0(a''(x_0))\leq|i_{A,\beta}(B_1)-i_{A,\beta}(B_0)|$. By Lemma 3.2 and Propositions 5.1 and 5.2, the proof is complete. ∎

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